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Animadversiones in rectificationem ellipsis

Leonhard Euler

Animadverſiones
in
Rectificationem Ellipſis.

§. I.

Tab. I.

Ellipſis rectificatio tot jam variis methodis eſt fruſtra tentata, ut non ſolum comparationem arcuum ellipticorum cum lineis rectis, ſed etiam ne cum circularibus quidem aut parabolicis expectare nequeamus. Cum enim formula illa differentialis, cujus integrale arcum ellipticum indefinitum exprimit, nullo modo ab irrationalitate liberari queat; certum hoc eſt ſignum, ejus integrationem non ſolum non algebraice, ſed etiam ne conceſſis quidem circuli & hyperbolæ quadraturis perfici poſſe. Quod cum tenendum ſit de rectificatione ellipſis indefinita, hinc adhuc minime ſequitur, arcum quempiam definitum veluti totam perimetrum ellipſis omnem comparationem cum lineis vel rectis vel circularibus penitus reſpuere: propterea quod jam innumerabiles curvæ aſſignari poſſunt indefinite æque parum rectificabiles atque ellipſis, in quibus tamen arcus definiti per lineas rectas menſurari queant.

§. II. Miſſa igitur rectificatione ellipſis indefinita, definitam potius ſum aggreſſus, experturus, utrum tota cujusque ellipſis perimeter non commode poſſit ad menſuras cognitas, quorſum etiam logarithmos & arcus circulares refero, per expreſſiones finitas revocari. Quanquam autem in hac inveſtigatione nihil admodum ſum conſecutus, quod ſcopo meo ſatisfeciſſet; tamen præter expectationem nonnulla ſe mihi obtulerunt phænomena ſatis ſingularia, quibus theoria linearum curvarum non mediocriter promoveri videtur. Tum vero etiam difficultates, quae in toto hoc calculo occurrerunt, anſam mihi praebuerunt quaedam inſignia artificia inveniendi, quae tam in calculo integrali, quam in theoria ſerierum infinitarum ingentem utilitatem ſaepius afferre poſſe videntur. Quamobrem operae pretium fore exiſtimavi, ſi has ſpeculationes totumque quaſi filum calculorum meorum dilucide expoſuero.

Propoſitio.

Fig. 1. *§. III. Super data recta AC tanquam altero ſemiaxe deſcriptos concipio infinitos quadrantes ellipticos AP, AB, Ap, quorum ergo omnium eſt centrum C, alteri vero ſemiaxes conjugati ſunt CP, CB, Cp. Tum ex ſingulis punctis P, B, p arcus elliptici PA, BA, pA in directum extendantur, itaut quaelibet PQ ſit rectae CA parallela & quadranti elliptico PTA aequalis: quod ſi ubique fieri concipiatur, puncta haec Q ſita erunt in linea quadam curva $AQDp$, cujus naturam inveſtigare conſtitui.*

Ad geneſin hujus curvae vel leviter attendenti mox patebit, eam ſequentes habere proprietates, quas evolvam, antequam in ipſam hujus curvae indolem diligentius inquiram; ut ejus figura & ductus ſaltem obiter perſpiciatur.

§. IV. Primum igitur ſi in recta indefinita CBp quae ad datam CA eſt normalis, capiatur quaevis abſciſſa CP, applicata PC, quae ei reſpondet, erit aequalis quadranti perimetri ellipſis, cujus ſemiaxes conjugati ſunt, recta data CA & ipſa abſciſſa CP. Hinc ſi capiatur abſciſſa $CB = CA$, quo caſu quadrans ellipticus abibit in quadrantem circularem AB, applicata reſpondens BD aequalis erit quartae parti peripheriae circuli radio AC deſcripti. Unde ſi ratio diametri ad peripheriam ponatur $= 1 : \pi$, erit iſta applicata

BD

$BD = \frac{1}{2}\pi . AC$: sive ob $\pi = 3,14159265358979 32$ erit $BD = 1,5707963267948961 . AC$.

§. V. Secundo: Si abscissa CP evanescat, ellipsis evadet infinite angusta, atque cum linea recta confundetur. Hoc ergo casu quadrans ellipticus abibit in ipsam lineam AC, cui propterea applicata abscissæ evanescenti respondens erit æqualis. Quare ipsa recta CA erit applicata puncto C respondens, & curva quæsita per punctum A transibit. Hujus ergo curvæ jam duo habemus puncta cognita A & D, quorum alterum A geometrice datur, alterum vero D per rationem diametri ad peripheriam definitur.

§. VI. Tertio: Ex cognito quovis curvæ puncto Q intra A & D sito, semper aliud quoddam curvæ punctum q ultra D situm definiri potest. Capiatur enim Cp tertia proportionalis ad CP & CA, ut sit $Cp = \frac{CA.CA}{CP.}$, quia est $CP : CA = CA : Cp$, erit quadrans ellipticus Ap similis quadranti elliptico AP, cum utrinque eadem sit ratio inter semiaxes conjugatos. Hinc erit arcus Ap ad arcum AP ut AC ad CP, ideoque $pq : PQ = AC : CP$ seu $pq = \frac{AC.PQ}{CP}$. Consequenter si curvæ quæsitæ arcus AD tantum jam fuerit descriptus, ex eo reliqua curvæ pars Dq in infinitum extensa definietur.

§. VII. Quarto: Hinc jam insignis proprietas æquationis, qua natura curvæ $AQDq$ exprimitur, agnoscitur. Si enim recta data AC unitate designetur, ut sit $AC = 1$; abscissa autem quævis unitate minor $CP = p$, eique respondens applicata $PQ = q$; tum vero ponatur abscissa illa altera $Cp = P$ & applicata $pq = Q$; erit $P = \frac{1}{p}$ & $Q = \frac{q}{p}$. Quare cum inter P & Q eadem esse debeat æquatio quæ est inter p & q, patet æquationem inter p & q nullam mutationem esse subituram, si in ea loco p ubique scribatur $\frac{1}{p}$ & $\frac{q}{p}$ loco q. Unde qualis ipsius p functio sit q conjicere licet.

§. VIII. Quinto: Patet crescentibus abscissis CP applicatas continuo crescere, cum semper sint majores quam abscissæ. Verum si abscissæ statuantur infinitæ, applicatæ ipsis fient æquales: discrimen enim prodibit infinite parvum; unde colligimus quæsitam curvam habere asymtotam, & quidem rectam CV angulum rectum ACB bisecantem. Forma igitur hujus curvæ similis erit hyperbolæ æquilateræ centrum in C, axem CA & asymtotam CV habentis. Ex descriptione porro intelligitur, curvam infra rectam CA productam sui similem fore, ideoque rectam CA ejus fore diametrum perinde atque hyperbolæ. Verumtamen hoc facile perspicitur, nostram curvam multo lentius ad asymtotam suam CV appropinquare quam hyperbolam. Nam in hyperbola æquilatera, cui nostram curvam comparamus, quævis applicata PQ æqualis est rectæ lineæ AP; unde cum applicata nostræ curvæ arcui AP sit æqualis, patet hyperbolam nostræ curvæ fore circum scriptam, ita tamen ut in initio A, & in spatio infinito se mutuo tangant.

§. IX. His affectionibus latius patentibus in genere notatis, in ipsam hujus curvæ naturam accuratius inquiramus, ac proposita quacunque abscissa CP $= p$, valorem respondentis applicatæ PQ $= q$ investigemus; qui cum expressione finita contineri nequeat, per seriem infinitam exhiberi debebit. Sequens igitur resolvi oportet

Pro-

Problema.

Ex datis semiaxibus CA & CP quadrantis elliptici CAP per seriem infinitam definire longitudinem arcus quadrantis ATP.

Solutio.

§. X. Cum vocatus sit alter semiaxis $AC = 1$, alter vero $CP = p$, & arcus $AYP = q$, quæratur primo arcus quivis indefinitus PY, qui vocetur $= s$. Jam ducta ad CP applicata normali YX, sit $CX = x$ & $XY = y$, erit ex natura ellipsis $x = p\sqrt{(1 - yy)}$, hincque $dx = \frac{-pydy}{\sqrt{(1-yy)}}$. Fiet ergo ob $ds = \sqrt{(dx^2 + dy^2)}$

$$ds = \frac{dy\sqrt{(1 - yy + ppyy)}}{\sqrt{(1 - yy)}}$$

unde integrando erit arcus $s = \int \frac{dy\sqrt{(1 - yy + ppyy)}}{\sqrt{(1 - yy)}}$.

quæ integratio ita institui debet, ut posito $y = 0$ fiat quoque $s = 0$, quia evanescente applicata $XY = y$ simul $PY = s$ evanescit. Hoc igitur integrali invento si ponatur $y = CA = 1$, arcus indefinitus PY abibit in longitudinem quadrantis elliptici $PYA = q$, quem quærimus, ita ut sit

$$q = \int \frac{dy\sqrt{(1 - yy + ppyy)}}{\sqrt{(1 - yy)}}$$

siquidem peracta integratione ponatur $y = 1$.

§. XI. Ad institutum ergo nostrum non est necesse, ut quæramus valorem integralis hujus indefiniti, sed eum tantum, quem induit, si post integrationem variabili y tribuatur valor determinatus $= 1$: quo pacto series multo simplicior valorem q exprimens obtineri poterit. Ponatur enim brevitatis gratia

$1 - pp = nn$, ut fit $\sqrt{(1 - yy + ppyy)} = \sqrt{(1 - nnyy)}$ eritque hanc formulam in seriem evolvendo.

$$\sqrt{(1-nnyy)} = 1 - \tfrac{1}{2} nnyy - \frac{1.1}{2.4} n^4 y^4 - \frac{1.1.3}{2.4.6} n^6 y^6 - \&c.$$

Quo valore substituto pro $\sqrt{(1 - yy + ppyy)}$, arcus q ita exprimetur ut sit:

$$q = \int \frac{dy}{\sqrt{(1-yy)}} - \tfrac{1}{2} nn \int \frac{yydy}{\sqrt{(1-yy)}} - \frac{1.1}{2.4} n^4 \int \frac{y^4 dy}{\sqrt{(1-yy)}}$$
$$- \frac{1.1.3}{2.4.6} \int \frac{y^6 dy}{\sqrt{(1-yy)}} \&c.$$

si quidem in singulis his integralibus post integrationem ponatur $y = 1$.

§. XII. Evolvamus ergo singula hæc integralia; ac primo quidem ex circulo manifestum est, formulam $\int \frac{dy}{\sqrt{(1-yy)}}$ exprimere arcum circuli cujus sinus $= y$ pro radio $= 1$: unde posito $y = 1$, hæc formula dabit quartam peripheriæ partem, cujus radius $= 1$. Ideoque posita ratione diametri ad peripheriam $= 1 : \pi$, erit $\int \frac{dy}{\sqrt{(1-yy)}} = \frac{\pi}{2}$; sicque jam adepti sumus valorem primi termini in serie nostra ante inventa.

§. XIII. Reliqui termini pari modo per valorem π commode poterunt exprimi; cujusvis enim termini integratio ad integrationem præcedentis reducitur: quod quo facilius intelligatur consideremus formulam quamcunque $\int \frac{y^\mu dy}{\sqrt{(1-yy)}}$; erit sequens

sy-

$\int \frac{y^{\mu+2}\,dy}{\sqrt{(1-yy)}}$. Jam assumamus hanc formulam algebraicam $y^{\mu+1}\sqrt{(1-yy)}$, cujus differentiale cum sit $=$

$$\frac{(\mu+1)y^{\mu}\,dy-(\mu+2)y^{\mu+2}\,dy}{\sqrt{(1-yy)}}$$ erit vicissim

$$(\mu+1)\int\frac{y^{\mu}\,dy}{\sqrt{(1-yy)}}-(\mu+2)\int\frac{y^{\mu+2}\,dy}{\sqrt{(1-yy)}}=y^{\mu+1}\sqrt{(1-yy)}$$

unde colligimus fore

$$\int\frac{y^{\mu+2}\,dy}{\sqrt{(1-yy)}}=\frac{\mu+1}{\mu+2}\int\frac{y^{\mu}\,dy}{\sqrt{(1-yy)}}-\frac{1}{\mu+2}y^{\mu+1}\sqrt{(1-yy)}$$

Quare invento integrali $\int\frac{y^{\mu}\,dy}{\sqrt{(-yy)}}$ ex eo facile elicitur integrale sequens $\int\frac{y^{\mu+2}\,dy}{\sqrt{(1-yy)}}$

§. XIV. Quoniam vero eos tantum horum integralium valores desideramus, qui prodeunt posito $y=1$; hoc casu quantitas algebraica $\frac{1}{\mu+1}\cdot y^{\mu+1}\sqrt{(1-yy)}$ evanescit, eritque generatim pro casu $y=1$

$\int y$

$$\int \frac{y^{\mu+2}\, dy}{\sqrt{(1-yy)}} = \frac{\mu+1}{\mu+2} \int \frac{y^{\mu}\, dy}{\sqrt{(1-yy)}}$$

substituamus jam pro μ successive valores 0, 2, 4, 6, 8 &c. & quoniam vidimus esse $\int \frac{dy}{\sqrt{(1-yy)}} = \frac{\pi}{2}$, erit, ut sequitur:

$$\text{si } \mu = 0;\ \int \frac{y^2\, dy}{\sqrt{(1-yy)}} = \frac{1}{2} \int \frac{dy}{\sqrt{(1-yy)}} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\mu = 2;\ \int \frac{y^4\, dy}{\sqrt{(1-yy)}} = \frac{3}{4} \int \frac{y^2\, dy}{\sqrt{(1-yy)}} = \frac{1.3}{2.4} \cdot \frac{\pi}{2}$$

$$\mu = 4;\ \int \frac{y^6\, dy}{\sqrt{(1-yy)}} = \frac{5}{6} \int \frac{y^4\, dy}{\sqrt{(1-yy)}} = \frac{1.3.5}{2.4.6} \cdot \frac{\pi}{2}$$

$$\mu = 6;\ \int \frac{y^8\, dy}{\sqrt{(1-yy)}} = \frac{7}{8} \int \frac{y^6\, dy}{\sqrt{(1-yy)}} = \frac{1.3.5.7}{2.4.6.8} \cdot \frac{\pi}{2}$$

unde lex, qua sequentes progrediuntur, sponte elucet.

§. XV. Quodsi jam isti valores pro formulis integralibus, ex quibus longitudo quadrantis elliptici q constari inventus est, substituantur, reperietur

$$q = \frac{\pi}{2} - \frac{1}{2} nn \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1.1}{2.4} n^4 \cdot \frac{1.3}{2.4} \cdot \frac{\pi}{2} - \frac{1.1.3}{2.4.6} n^6 \cdot \frac{1.3.5}{2.4.6} \cdot \frac{\pi}{2}$$
&c.

quae ad sequentem seriem satis concinnam revocatur

$$q = \frac{\pi}{2}\left(1 - \frac{1.1}{2.2} n^2 - \frac{1.1.1.3}{2.2.4.4} n^4 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} n^6 - \&c.\right)$$

cujus

cujus lex progreſſionis eſt manifeſta. Reſtituatur ergo pro nn ſuus valor $1 - pp$, eritque

$$q = \frac{\pi}{2}\left(1 - \frac{1.1}{2.2}(1-pp) - \frac{1.1.1.3}{2.2.4.4}(1-pp)^2 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}(1-pp)^3 - \&c.\right)$$

§. XVI. Cum pro curva noſtra $AQDq$ littera p exhibeat abſciſſam CP & littera q applicatam PQ; jam adepti ſumus pro iſta curva æquationem inter ejus coordinatas p & q, quæ etſi ſerie conſtat infinita, tamen non ſolum ejus naturam in ſe complectitur, ſed etiam valores applicatæ q mox ſatis accurate exhibet, ſi abſciſſa p parum ab unitate differat: hoc eſt, cum ſit $CB = CA = 1$, ſi punctum P ipſi B fuerit proximum; tum enim ob $1 - pp = nn$ quantitatem valde parvam ſeries inventa valde convergit.

§. XVII. Hinc igitur indolem noſtræ curvæ prope punctum D, hoc eſt ejus directionem & curvaturam definire poterimus. Primo enim patet uti jam vidimus, ſi $p = 1$ fore $q = \frac{\pi}{2}$, ita ut ſumta abſciſſa $CB = 1$ ſit applicata $BD = \frac{\pi}{2} = 1,5707963267948961$. Deinde ad poſitionem tangentis inveniendam, quæratur ratio differentialium $dq : dp$, quæ per differentiationem reperitur:

$$\frac{dq}{dp} = \frac{\pi}{2}p\left(\frac{1}{2} + \frac{1.1.3}{2.2.4}(1-pp) + \frac{1.1.3.3.5}{2.2.4.4.6}(1-pp)^2 + \frac{1.1.3.3.5.5.7}{2.2.4.4.6.6.8}(1-pp)^3 + \&c.\right)$$

Posito jam $p = 1$ fiet $\frac{dq}{dp} = \frac{\pi}{4}$. Unde si DG sit tangens curvae in puncto D, cum sit BD: BG $= dq : dp$, erit BG $= \frac{dp}{dq}$ BD $= \frac{4}{\pi}$. BD & ob BD $= \frac{\pi}{2}$, fiet BH $= 2 = 2$BC, & CG $=$ BC. Sicque hoc casu subtangens BG erit dupla abscissae BC: & cum anguli BGD tangens sit $= \frac{dq}{dp} = \frac{\pi}{4} = 0,78539816$ erit angulus BGD $= 38^{\circ}, 8^{\mathrm{I}}, 45^{\mathrm{II}}, 41^{\mathrm{III}}, 51^{\mathrm{IV}}$.

§. XVIII. Ad radium osculi seu evolutae in puncto D definiendum, cum sit ob $\frac{dq}{dp} = \frac{\pi}{4}$, elementum curvae $\sqrt{(dp^2 + dq^2)} = dp\sqrt{(1 + \frac{\pi\pi}{16})}$, erit radius osculi $= (1 + \frac{\pi\pi}{16})^{3:2} dp^2 : ddq$.

At sumendis differentialibus secundis erit

$$\frac{ddq}{dp^2} = \frac{\pi}{2}\left(\frac{1}{2} + \frac{1.1.3}{2.2.4}(1 - pp) + \frac{1.1.3.3.5}{2.2.4.4.6}(1 - pp)^2 + \&c.\right)$$
$$- \frac{\pi}{2}\, pp\left(\frac{1.3}{2.4} + \frac{1.1.3.5}{2.2.4.6}(1 - pp) + \frac{1.3.3.5.5.7}{2.2.4.4.6.7}(1 - pp)^2 + \&c.\right)$$

Posito ergo $p = 1$, erit $\frac{ddq}{dp^2} = \frac{\pi}{2}\left(\frac{1}{2} - \frac{3}{8}\right) = \frac{\pi}{16}$: Unde in puncto curvae D erit radius evolutae $= \frac{16}{\pi}\left(1 + \frac{\pi\pi}{16}\right)\sqrt{\left(1 + \frac{\pi\pi}{16}\right)}$, qui valor in numeris proxime reperitur $= 10,470672$.

§. XIX.

§. XIX. Potest hinc adhuc alia series inveniri, quae valorem applicatae $PQ = q$ exprimat. Consideretur enim illud alterum curvae punctum q, pro quo sit abscissa $Cp = P$ & applicata $pq = Q$, erit quoque ob $P > 1$.

$$Q = \frac{\pi}{2}\left(1 + \frac{1.1}{2.2}(PP - 1) - \frac{1.1.1.3}{2.2.4.4}(PP-1)^2 + \frac{1.1.1.3.3.5}{2.2.4.4.6.6}(PP-1)^3 - \&c.\right)$$

Jam vero supra notavimus, si sit $P = \frac{1}{p}$, fore $Q = \frac{q}{p}$; quare his valoribus substitutis impetrabimus novam aequationem inter p & q, qua natura curvae pariter exprimetur.

$$q = \frac{\pi}{2} p \left(1 + \frac{1.1}{2.2}\frac{(1-pp)}{pp} - \frac{1.1.1.3}{2.2.4.4}\frac{(1-pp)^2}{p^4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6}\frac{(1-pp)^3}{p^6} - \&c.\right)$$

quae si cum ante inventa combinetur, innumerabiles aliae novae aequationes obtineri poterunt. Veluti si prior per p multiplicata ab hac subtrahatur, prodibit.

$$q - pq = \frac{\pi}{2} p \left(\frac{1.1}{2.2}\frac{(1-pp)(1+pp)}{pp} - \frac{1.1.1.3}{2.2.4.4}\frac{(1-pp)^2(1-p^4)}{p^4} + \&c.\right)$$

quae reducitur ad hanc:

$$q = \frac{\pi}{4}(1+p)\left(\frac{1}{2}\cdot\frac{1+pp}{p} - \frac{1.1.3}{2.4.4}\,\frac{(1-p^4)(1-pp)}{p^3} + \frac{1.1.3.3.5}{2.4.4.6.6}\,\frac{(1+p^6)(1-pp)^2}{p^5} - \&c.\right)$$

vel cum series adhuc sit divisibilis per $\frac{1+pp}{2p}$ erit

$$q = \frac{\pi}{8}\,\frac{(1+p)(1+pp)}{p}\left(1 - \frac{1.3}{4.4}\,\frac{(1-pp)}{pp}(1-pp) + \frac{1.3.3.5}{4.4.6.6}\,\frac{(1-pp+p^4)}{p^4}(1-pp)^2 - \frac{1.3.3.5.5.7}{4.4.6.6.8.8}\,\frac{(1-pp+p^4-p^6)}{p^5}(1-pp)^3 + \&c.\right)$$

§. XX. Manifestum autem est has series parum subsidii afferre, si applicatas invenire velimus, quae longius a BD, quae abscissae $p = 1$ respondet, sint remotae, si enim pro p ponatur numerus vel valde magnus vel valde parvus, series inventa vel parum admodum convergit vel etiam divergit. Si enim inde longitudinem primae applicatae CA, quae abcissae $p = 0$ respondet, definire velimus, serie primum inventa uti conveniet, quia in reliquis termini evadunt infinite magni. Habebimus igitur pro hoc casu $p = 0$;

$$q = \frac{\pi}{2}\left(1 - \frac{1.1}{2.2} - \frac{1.1.1.2}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.3} - \&c.\right)$$

quae tam lente convergit, ut etiamsi plurimi termini actu colligerentur, tamen verus ipsius q valor, quem novimus esse $= 1$, inde difficillime agnosci posset.

§. XXI.

§. XXI. Quanquam autem nunc quidem novimus esse

$$1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c. = \frac{2}{\pi}$$

tamen inventio summae hujus seriei non parum ardua videtur, si a priori tentetur. Veritatem quidem ex formula, quam quondam Wallisius pro circuli quadratura dedit, intelligere licet, si termini ab initio in unum colligantur, sic enim prodit

$$1 - \frac{1.1}{2.2} = \frac{1.3}{2.2}$$

$$\frac{1.3}{2.2} - \frac{1.1.1.3}{2.2.4.4} = \frac{1.3.(4.4 - 1.1)}{2.2.\ \ 4.4} = \frac{1.3.3.5}{2.2.4.4}$$

$$\frac{1.3.3.5}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} = \frac{1.3.3.5.5.7}{2.2.4.4.6.6}$$

unde valor seriei in infinitum continuatae erit

$$\frac{1.3.3.5.5.7.7.9.9.11.11.13.13}{2.2.4.4.6.6.8.8.10.10.12.12.14}\ \&c.$$

quae expressio cum sit ipsa Wallisiana, patet summam nostrae seriei esse $= \frac{2}{\pi}$. Interim tamen juvabit tradere methodum hanc seriem aliasque similes a priori summandi.

Problema.

Invenire summam hujus seriei infinitae:

$$1 - \frac{1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c.$$

cujus lex progressionis primo intuitu est manifesta.

Solutio.

§. XXII. Ponatur summa hujus seriei, quae quaeritur $= s$, ut sit

$$s = 1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c.$$

Jam eligatur series, cujus summa constat, & cujus coefficientes jam in his terminis contineantur. Cujusmodi est haec

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \tfrac{1}{2}xx + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \frac{1.3.5.7}{2.4.6.8}x^8 + \&c.$$

Erit ergo per differentiale quodpiam dP multiplicando & integrando

$$\int\frac{dP}{\sqrt{(1-xx)}} = P + \tfrac{1}{2}\int xx\,dP + \frac{1.3}{2.4}\int x^4\,dP + \frac{1.3.5}{2.4.6}\int x^6\,dP + \&c.$$

Nunc differentiale hoc dP ita definiatur, ut si post integrationem ponatur $x = 1$. fiat

$$\int x\,dP = -\tfrac{1}{2}P$$

$$\int x^4\,dP = +\tfrac{1}{4}\int xx\,dP = -\frac{1.1}{2.4}P$$

$$\int x^6\,dP = +\tfrac{3}{6}\int x^4\,dP = -\frac{1.1.3}{2.4.6}P$$

$$\int x^8\,dP = +\tfrac{5}{8}\int x^6\,dP = -\frac{1.1.3.5}{2.4.6.8}P$$

quo facto si hi valores substituantur habebitur;

$$\int\frac{dP}{\sqrt{(1-xx)}} = P\left(1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c.\right)$$

ideoque

ideoque $\int \frac{dP}{\sqrt{(1-xx)}} = P\iota$, ſi quidem poſt integrationem ſtatuatur $x = 1$.

§. XXIII. Huc ergo res redit, ut quæratur formula differentialis dP, ut ſuperioribus conditionibus ſatisfiat.

ſeu ut in genere fit $\int x^{\mu+2} dP = \frac{\mu-1}{\mu-2} \int x^{\mu} dP$, ſi quidem poſt integrationem utramque ponatur $x = 1$. Omiſſa igitur hac conditione fit

$$\int x^{\mu+2} dP = \frac{\mu-1}{\mu+2} \int x^{\mu} dP + \frac{Qx^{\mu+2}}{\mu+2}$$

ubi Q ejus modi ſit functio ipſius x, quæ evaneſcat poſito $x = 1$.

Capiantur ergo differentialia, eritque per x^{μ} dividendo

$$xxdP = \frac{\mu-1}{\mu+2}\, dP + \frac{xdQ + (\mu+1)\,Qdx}{\mu+2}$$

ſeu $0 = (\mu-1)\,dP - (\mu+2)\,xxdP + xdQ + (\mu+1)\,Qdx$ quæ æquatio, cum locum habere debeat pro omni valore ipſius μ, reſolvetur in has duas:

$$0 = dP - xxdP + Qdx$$

$$0 = -\,dP - 2xxdP + xdQ + Qdx$$

unde fit $dP = \frac{-Qdx}{1-xx} = \frac{xdQ + Qdx}{1+2xx}$

& $xdQ\,(1-xx) = -Qdx\,(2+xx)$

Quæ

Quare cum sit $\frac{dQ}{Q} = \frac{dx\,(2+xx)}{x\,(1-xx)} = -\frac{2\,dx}{x} - \frac{3\,xdx}{1-xx}$

erit $Q = \frac{-(1-xx)^{\frac{3}{2}}}{xx}$ & $dP = \frac{dx}{xx}\sqrt{(1-xx)}$

§. XXIV. Verum hic notandum est, etsi valor ipsius Q evanescat posito $x = 1$; tamen casu $\mu = 0$, quantitatem algebraicam $\frac{Qx^{\mu+1}}{\mu+2}$ non evanescere, si ponatur $x = 0$, quae tamen conditio aeque est necessaria atque altera, ita ut hoc casu non sit $\int xxdP = -\frac{1}{2}P$. Cum autem reliquae formulae quibus $\mu > 0$ locum habeant, a formula $\int xxdP$ erit incipiendum, eritque

$$\int x^4\,dP = \frac{1}{4}\int xxdP$$

$$\int x^6\,dP = \frac{3}{6}\int x^4\,dP = \frac{1.3}{4.6}\int xxdP$$

$$\int x^8\,dP = \frac{5}{8}\int x^6\,dP = \frac{1.3.5}{4.6.8}\int xxdP$$

&c.

unde habebitur

$$\int\frac{dP}{\sqrt{(1-xx)}} = P + \int xxdP\left(\frac{1}{2} + \frac{1.1.3}{2.4.4} + \frac{1.1.3.3.5}{2.4.4.6.6} + \&c.\right)$$

At est $\frac{1}{2} + \frac{1.1.3}{2.4.4} + \frac{1.1.3.3.5}{2.4.4.6.6} + \& = 2\,(1 - l)$; ideoque

$\int dP$

$\int \frac{dP}{\sqrt{(1-xx)}} = P + 2(1-s)\int xxdP$. At ob $dP = \frac{dx}{xx}\sqrt{(1-xx)}$

erit $P = C - \frac{\sqrt{(1-xx)}}{x} - A \sin x$;

$$\int xxdP = \int dx \sqrt{(1-xx)} = \frac{1}{2} A \sin x + \frac{1}{2} x\sqrt{(1-xx)}, \& \int \frac{dP}{\sqrt{(1-xx)}} = D - \frac{1}{x}$$

ubi constantes C & D ita accipi debent, ut integralia hæc evanescant posito $x = 0$: quanquam autem utraque seorsim sit infinita, tamen conjunctæ se mutuo destruent. Erit enim

$$\int \frac{dP}{\sqrt{(1-xx)}} - P = D - \frac{1}{x} - C + \frac{\sqrt{(1-xx)}}{x} + A \sin x$$

quæ ut evanescat posito $x = 0$, debet esse $D = C$, ideoque posito jam $x = 1$ fiet $\int \frac{dP}{\sqrt{(1-xx)}} - P = -1 + \frac{\pi}{2}$: & quia eodem hoc casu est $\int xxdP = \frac{\pi}{4}$, prodibit

$$-1 + \frac{\pi}{2} = 2(1-s)\frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{2}s$$

hincque colligitur fore $\frac{\pi}{2}s = 1$ & $s = \frac{2}{\pi}$ seu

$$1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \& = \frac{2}{\pi}$$

uti ex rei natura jam conclusimus.

§. XXV. Quoniam igitur eruimus in ipso initio esse applicatam curvæ $CA = 1$, indolem hujus curvæ prope punctum A indagemus, seu in valorem applicatæ q inquiramus, si abscissa p fuerit valde parva. In hunc finem ponamus iterum $1 - pp = nn$, & cum sit

$$q = \frac{\pi}{2}\left(1 - \frac{1.1}{2.2}nn - \frac{1.1.1.3}{2.2.4.4}n^4 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}n^6 - \&c.\right)$$

& quia novimus fore praxime $q = 1$, addamus æqualitatem modo inventam:

$$0 = 1 - \frac{\pi}{2}\left(1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}n^6 - \&c.\right)$$

atque habebimus:

$$q = 1 + \frac{\pi}{2}\left(\frac{1.1}{2.2}(1 - nn) + \frac{1.1.1.3}{2.2.4.4}(1 - n^4) + \frac{1.1.1.3.3.5}{2.2.4.4.6.6}(1 - n^6) + \&c.\right)$$

cujus seriei cum singuli termini sint per $1 - nn = pp$ divisibiles, reducetur hæc expressio ad hanc:

$$q = 1 + \frac{\pi}{8}pp\left(1 + \frac{1.3}{4.4}(1 + nn) + \frac{1.3.3.5}{4.4.6.6}(1 + nn + n^4) + \frac{1.1.3.3.5.5.7}{4.4.6.6.8.8}(1 + n^2 + n^4 + n^6) + \&c.\right)$$

§. XXVI. Quodsi in hac expressione singuli termini ad potestates ipsius n evolvantur, reperietur

+

$$q = 1 + \frac{\pi}{2}pp \left\{ \begin{array}{l} + \frac{1.1}{2.2} + \frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \\ \quad \frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8} + \&c. \\ + n^2 \left(\frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \right. \\ \quad \frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8} + \&c. \\ + n^4 \left(\frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8} + \right. \\ \quad \&c. \\ + n^6 \left(\frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8} + \&c. \right. \\ \quad \&c. \end{array} \right.$$

At ex supra inventis habemus summam primæ seriei

$$\frac{1.1}{2.2} + \frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \&c. = 1 - \frac{2}{\pi}$$

quæ si primo termino multetur, prodibit secunda, quæ est coefficiens ipsius nn, ita ut sit

$$\frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \&c. = \frac{1.3}{2.2} - \frac{2}{\pi}$$

hæc denuo primo termino multata, dabit coefficientem ipsius n^4, nempe

$$\frac{1.1.1.3.3.5}{2.2.4.4.6.6} + \&c. = \frac{1.3.3.5}{2.2.4.4} - \frac{2}{\pi}$$

similique modo coefficiens ipsius n^6 erit $= \frac{1.3.3.5.5.7}{2.2.4.4.6.6} - \frac{2}{\pi}$ & ita porro, sicque tandem obtinebitur.

$$q = 1 + \frac{\pi}{2}\, pp\Big((1 - \frac{2}{\pi}) + (\frac{1.3}{2.2} - \frac{2}{\pi})\, nn + (\frac{1.3.3.5}{2.2.4.4} - \frac{2}{\pi})\, n^4 + (\frac{1.3.3.5.5.7}{2.2.4.4.6.6} - \frac{2}{\pi})\, n^6 + \&c. \Big)$$

vel erit;

$$q = 1 + pp\Big((\frac{\pi}{2} - 1) + (\frac{1.3}{2.2}\,\frac{\pi}{2} - 1)\, nn + (\frac{1.3.3.5}{2.2.4.4}\,\frac{\pi}{2} - 1)\, n^4 + (\frac{1.3.3.5.5.7}{2.2.4.4.6.6}\,\frac{\pi}{2} - 1)\, n^6 + \&c. \Big)$$

§. XXVII. Ponamus jam hic $n = 1$, ut obtineamus aequationem hujus formae $q = 1 + App$, qua natura curvae prope punctum A exprimitur: cum enim conjectare liceat veram aequationem futuram esse hujus formae:

$$q = 1 + App + Bp^4 + Cp^6 + Dp^8 + \&c.$$

si abscissa p valde parva sumatur, reliqui termini praeter binos primos ommitti poterunt, atque ex aequatione $q = 1 + App$, tam positio tangentis, quam curvatura in puncto A colligi poterit. Posito enim $AR = x$, $RQ = y$, erit $q = 1 + x$ & $p = y$, ideoque si arcus AQ fuerit minimus, is cum parabola confundetur, cujus aequatio $x = Ayy$ seu $yy = \frac{1}{A}x$, ac propterea $\frac{1}{A}$ parameter. Unde sequitur tangentem curvae in A fore ad rectam AC perpendicularem, & radium osculi ibidem esse $= \frac{1}{2A}$.

§. XXVIII.

§. XXVIII. Hic igitur coefficiens A reperietur, si in superiore serie, per quam quantitas pp multiplicatur, ponatur $n = 1$; ita ut sit

$$A = \left(\frac{\pi}{2} - 1\right) + \left(\frac{1.2}{2.2} \cdot \frac{\pi}{2} - 1\right) + \left(\frac{1.3.3.5}{2.2.4.4} \cdot \frac{\pi}{2} - 1\right) + \&c.$$

quae autem si ejus summatio tentetur, tam parum convergens deprehenditur, ut ejus summam adeo infinitam suspicari debeamus. In hac autem suspicione eo magis confirmamur, si seriem primo (§. 15.) inventam, secundum dimensiones ipsius p evolvamus, unde fit

$$= \frac{\pi}{2} \begin{cases} 1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c. \\ + p_2 \left(\frac{1.1}{2.2} + \frac{1.1.1.2}{2.2.4.4} 2 + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} \cdot 3 + \&c.\right) \\ - p_4 \left(\frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} \cdot 3 + \&c.\right) \end{cases}$$

§. XXIX. Hinc ergo coefficiens ipsius pp in aequatione generali pro curva $q = 1 + App + Bp^4 + Cp^6 + Dp^8 + \&c.$ erit

$$A = \frac{\pi}{2} \left(\frac{1.1}{2.2} \cdot 1 + \frac{1.1.1.3}{2.2.4.4} \cdot 2 + \frac{1.1.1.3.3.6}{2.2.4.4.6.6} \cdot 3 + \&c.\right)$$

$$\text{seu } A = \frac{\pi}{4} \left(\frac{1}{2} + \frac{1.1.3}{2.2.4} + \frac{1.1.3.3.5}{2.2.4.4.6} + \frac{1.1.3.3.5.5.7}{2.2.4.4.6.6.8} + \&c.\right)$$

similique modo & reliquos coefficientes B, C, D. &c. ex hac serie eruere licebit. Verum hoc labore supersedere poterimus, cum liqueat non solum coefficientem A, sed etiam omnes reliquos prodituros

dituros esse infinitos. Perspicuum hoc fiet ex solutione hujus problematis.

Problema.

Invenire summam hujus seriei infinitæ:

$$S = \frac{1}{2} + \frac{1.1.3}{2.2.4} + \frac{1.1.3.3.5}{2.2.4.4.6} + \&c.$$

Solutio.

§. XXX. Assumatur ad hanc summam s inveniendam hæc formula:

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \tfrac{1}{2}xx + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \&c.$$

ut sit

$$\int \frac{dP}{\sqrt{(1-xx)}} = P + \tfrac{1}{2}\int xxdP + \frac{1.3}{2.4}\int x^4 dP + \frac{1.3.5}{2.4.6}\int x^6 dP + \&c.$$

fitque si post integrationes singulas ponatur $x = 1$

$$\int xxdP = \frac{3}{4}P$$

$$\int x^4 dP = \frac{5}{6}\int xxdP = \frac{3.5}{4.6}P$$

$$\int x^6 dP = \frac{7}{8}\int x^4 dP = \frac{3.5.7}{4.6.8}P$$

&c.

hincque fiet

$\int dP$

$$\frac{dP}{\sqrt{(1-xx)}} = P\left(1+\frac{1.3}{2.3}+\frac{1.3.3.5}{2.4.4.6}+\frac{1.3.3.5.5.7}{2.4.4.6.6.8}+\&c.\right)$$

five $\int \frac{dP}{\sqrt{(1-xx)}} = 2Pr$; unde invento P, reperietur r si post integrationem ponatur $x = 1$.

§. XXXI. Cum igitur generaliter esse debeat.

$$\int x^{\mu+1} dP = \frac{\mu+3}{\mu+4}\int x^{\mu} dP + \frac{x^{\mu+1} Q}{\mu+4}$$

dummodo Q ejusmodi sit functio, quæ evanescat posito $x = 1$, erit

$$(\mu+4)\, xxdP = (\mu+3)\, dP + xdQ + (\mu+1) Qdx$$

unde duæ sequentes æquationes conficiuntur.

$$xxdP = dP + Qdx$$

$$4xxdP = 3dP + dQ + Qdx$$

& $dP = \frac{-Qdx}{1-xx} = \frac{-xdQ-Qdx}{3-4xx}$

hincque elicitur $\frac{dQ}{Q} = \frac{2dx-3xxdx}{x(1-xx)} = \frac{2dx}{x} \quad \frac{-xdx}{1-xx}$

& $Q = -xx\sqrt{(1-xx)}$. Quare habebitur

$$dP = \frac{xxdx}{\sqrt{(1-xx)}} \;\&\; \frac{dP}{\sqrt{(1-xx)}} = \frac{xxdx}{1-xx} = -dx + \frac{dx}{1-xx}.$$

Fiet ergo $P = \frac{3}{4}r$, si post integrationem ponatur $x = 1$

et

at $\int \frac{dP}{\sqrt{(1-xx)}} = -x + \frac{1}{2} l \frac{1+x}{1-x}$, cujus valor posito $x = 1$ fit utique infinitus. Erit igitur $s = \infty$, seu summa seriei propositæ infinite magna.

§. XXXII. Quia igitur coefficiens A ipsius pp, in æquatione

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + \&c.$$

est infinitus, radius osculi curvæ in puncto A utique erit infinite parvus. Verum præterea hæc æquatio: in qua omnes omnino coefficientes A, B, C, D &c. sunt infiniti, nihil plane ad curvæ cognitionem confert. Quia enim radius osculi curvæ in A est infinite parvus, natura curvæ circa punctum A hujus modi æquatione $q = 1 + \alpha p^m$ exprimetur, in qua exponens m binario sit minor, verumtamen unitate major: sed ex omnibus, quæ hactenus sunt tradita nulla via patet, qua hunc exponentem m scrutari queamus. Cum enim is numerus integer esse nequeat, nulla seriorum, quas pro q eruimus, ita est comparata, ut ex ea potestatem ipsius p irrationalem elicere liceat.

§. XXXIII. Hinc intelligimus problema esse summopere difficile, quo æquatio tantum elementaris requiritur, quæ naturam curvæ propositæ AQDq saltem proxime circa punctum A exhibeat. Notum est enim si ponatur AR $= x$ & RQ $= y$, quæcunque fuerit curva AQ, naturam minimæ ejus portiunculæ circa A semper hujusmodi æquatione $y^m = Ax$, comprehendi posse; siquidem curva sit algebraica; pro curvis autem transcendentibus certum videtur, quasvis earum minimas portiunculas cum arcubus curvarum algebraicarum comparari posse. Quare in nostra curva,

etsi

etsi est transcendens, hoc eo magis mirum videri debet, quod nulla hujusmodi formula $y = Ax^m$ exhiberi possit, quae saltem minimae ejus portiunculae circa A sitae naturam declaret.

§. XXXIV. Hunc nodum ut resolvamus, aequationem nobis finitam inter coordinatas p & q investigare oportebit, quae etsi, ut facile praevidere licet, ad differentialia secundi ordinis exsurget, tamen ad accuratiorem curvae cognitionem magis erit accomodata. Eliciemus autem hujusmodi aequationem, quae numero terminorum finito constet, si seriem primo inventam ad summam revocabimus. Cum enim posito $1 - pp = nn$ sit

$$\frac{2q}{\pi} = 1 - \frac{1.1}{2.2}nn - \frac{1.1.1.3}{2.2.4.4}n^4 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}n^6 - \&c.$$

Erit differentiando

$$\frac{2dq}{\pi dn} = -\frac{1.1}{2}n - \frac{1.1.1.3}{2.2.4}n^3 - \frac{1.1.1.3.3.5}{2.2.4.4.6}n^5 - \&c.$$

quae per n multiplicata denuoque differentiata dat

$$\frac{2}{\pi dn}\,d\,\frac{ndq}{dn} = -1.1n - \frac{1.1}{2.2}1.3n^3 - \frac{1.1.1.3}{2.2.4.4}3.5n^5 - \&c.$$

Multiplicetur haec per $\frac{dn}{n}$ ac rursus integretur, erit

$$\frac{2}{\pi}\int\frac{1}{n}\,d.\frac{ndq}{dn} = -1.n - \frac{1.1}{2.2}.1n^3 - \frac{1.1.1.3}{2.2.4.4}.3n^5 - \&c.$$

Multiplicetur per $\frac{dn}{n^3}$ & integrando prodibit,

$$\frac{2}{\pi}\int\frac{dn}{n}\int\frac{1}{n}\,d.\,\frac{ndq}{dn} = \frac{1}{n} - \frac{1.1}{2.2}\,n - \frac{1.1.1.3}{2.2.4.4}\,n^3 - \&c.$$

quae series cum sit ipsa proposita, per n divisa erit

$$\frac{2}{\pi}\int\frac{dn}{nn}\int\frac{1}{n}\,d.\,\frac{ndq}{dn} = \frac{2q}{\pi n} \text{ seu } \int\frac{dn}{nn}\int\frac{1}{n}\,d.\,\frac{ndq}{dn} = \frac{q}{n}$$

§. XXXV. Sumamus nunc differentialia, habebiturque

$$\frac{dn}{n^2}\int\frac{1}{n}\,d.\,\frac{ndq}{dn} = \frac{ndq - qdn}{nn} \text{ seu}$$

$$\int\frac{1}{n}\,d.\,\frac{ndq}{dn} = \frac{nndq}{dn} - nq.$$

porroque differentiando

$$\frac{1}{n}\,d.\,\frac{ndq}{dn} = nd.\,\frac{ndq}{dn} + ndq - ndq - qdn$$

$$\text{seu } (1-nn)\,d.\,\frac{ndq}{dn} + qndn = 0$$

Jam ob $1-nn = pp$ erit $ndn = -pdp$ & $\frac{dn}{n} = -\frac{pdp}{1-pp}$

unde fit $-ppd.\,\frac{(1-pp)\,dq}{pdp} - pqdp = 0$ seu

$d.\,\frac{(1-pp)\,dq}{pdp} + \frac{qdp}{p} = 0.$ Sumatur jam dp constans erit

$$\frac{(1-pp)\,dq}{pdp} - \frac{dq\,(1+pp)}{pp} + \frac{qdp}{p} = 0 \text{ seu}$$

$$p\,(1-pp)\,ddq - dpdq\,(1+pp) + pqdp^2 = 0.$$

§. XXXVI.

§. XXXVI. En igitur aequationem differentialem secundi gradus pro curva proposita

$$p\,(1 - pp)\,ddq - dpdq\,(1 + pp) + pqdp^2 = 0$$

ex qua potestas illa ipsius p in aequatione $q = 1 + Ap^m$ elici debet, si abscissa p valde parva statuatur. Cum igitur sit $dq = m\,Ap^{m-1}\,dp$ & $ddq = m\,(m-1)\,Ap^{m-2}\,dp^2$ oritur

$$\left.\begin{array}{l} m\,(m-1)\,Ap^{m-1} - mAp^{m-1} + p \\ -m\,(m-1)\,Ap^{m+1} - m\,Ap^{m+1} + Ap^{m+1} \end{array}\right\} = 0$$

seu $m\,(m-2)\,Ap^{m-1} - mmAp^{m+1} + p = 0.$

Deberet ergo esse $m = 2$, ut terminus Ap^{m-1} cum p comparari posset, sed tum iterum obtinetur $A = \infty$: praeterea vero hinc perspicitur exponentem m nullo modo numerum fractum esse posse, ita ut hinc difficultas supra memorata augeri potius quam tolli videatur.

§. XXXVII. Quodsi regulis consuetis uti velimus ad aequationem inventam in seriem evolvendam, quae secundum potestates ipsius p procedat, quoniam novimus primum seriei terminum esse $= 1$, nullam aliam formam inde colligere licet nisi hanc:

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$$

unde fit

$$\frac{dq}{dp} = 2Ap + 4Bp^3 + 6Cp^5 + 8Dp^7 + \&c.$$

& $\frac{ddq}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \&c.$

qui valores in æquatione ſubſtituti præbebunt:

$$\left.\begin{array}{rrrrr} +2Ap & +12Bp^3 & +30Cp^5 & +56Dp^7 & +\&c. \\ & -2A & -12B & -30C & -\&c. \\ -2Ap & -4B & -6C & -8D & -\&c. \\ & -2A & -4B & -6C & -\&c. \\ +p & +A & +B & +C & +\&c. \end{array}\right\} = 0$$

unde omnes coefficientes A, B, C, &c. prodeunt infiniti.

§. XXXVIII. Hinc igitur videmus regulas ordinarias, ſecundum quas vulgo forma ſeriei, in quam æquatio differentialis transmutanda ſit, dijudicari ſolet, non eſſe ſufficientes, cum hoc caſu nullam afferant utilitatem: unde noſtra æquatio eo majorem meretur attentionem. Sequenti tamen modo ex ea natura curvæ prope punctum A colligi poterit, ex quo ſimul intelligetur, quemadmodum quoque in aliis caſibus defectus iſte regularum uſu receptarum ſuppleri, eæque ad praxin accommodari debeant. Quia enim abſciſſam p hic pro infinite parva habemus, in æquatione pro $1 - pp$ & $1 + pp$ ponere licebit 1, & quia novimus eſſe hoc caſu proxime $q = 1$, pro quantitate finita q unitatem ſcribamus: quo facto æquatio differentio-differentialis inventa pro caſu, quo abſciſſa p eſt minima ſequentem induet formam;

$$pddq - dpdq + pdp^2 = 0.$$

§. XXXIX. Hujus jam æquationis reſolutio eſt facilis, cum enim dp ſit conſtans, ponatur $dq = rdp$, erit $ddq = drdp$, habebiturque:

$$pdr - rdp + pdp = 0$$

pdr-five $\frac{pdr - rdp}{pp} + \frac{dp}{p} = 0$

cujus integrale eft: $\frac{r}{p} + lp = C$, unde fit

$$r = Cp - plp$$

ideoque $dq = Cpdp - pdp\ lp$

Hæc jam æquatio integrata dabit:

$$q = 1 + \tfrac{1}{2} Cp^2 - pp\ lp + \tfrac{1}{4} pp$$

in qua cum terminus pp incomparabiliter fit minor quam $pp\ ld$, erit pro curvæ initio A:

$$q = 1 - \tfrac{1}{2}\ pp\ lp$$

§. XL. Nuncigitur naturam curvæ prope initium A æquatione fimplici definire poffumus: fi enim vocemus AR $= x$ & RQ $= y$, ob $p = y$ & $q = 1 + x$, orietur hæc $x = -\frac{1}{2}\ yyly$, ad quam æquatio generalis pro curva revocatur, fi coordinatæ x & y fint quam minimæ. Patet igitur ne minimum quidem arculum circa A tanquam portiunculam curvæ algebraicæ fpectari poffe, fed ejus naturam logarithmos implicare. Et quoniam æquatio logarithmica in exponentialem transformari poteft, initium curvæ noftræ A commune erit cum linea tranfcendente, cujus æquatio eft $e^{-2x} = y^{yy}$, fumto e pro numero cujus logarithmus hyperbolicus eft $= 1$.

§. XLI. Æquatione hac $x = -\frac{1}{2} yyly$ confirmantur quoque ea, quæ fupra jam de affectionibus hujus curvæ in puncto A notavimus. Primo enim patet fi fit $y = 0$, fore quoque $yyly$ ac proinde $x = 0$, etfi hoc cafu fit $ly = -\infty$. Deinde cum fit $dy = -y\ ly\ ly - \frac{1}{2}\ y\ ly$, quia y incomparabiliter eft minus quam yly,

 erit

erit $dx = - y\,dy\,ly$, ac propterea $\frac{dy}{dx} = \frac{-1}{yly} = \infty$ posito $y = 0$; unde patet tangentem curvæ in A ad abscissam A R esse perpendicularem. Porro cum sit subnormalis $\frac{ydy}{dx} = \frac{-1}{ly}$, hocque casu subnormalis radio evolutæ æquetur, ob $ly = \infty$ si $y = 0$, manifestum est radium osculi curvæ in A esse infinite parvum.

§. XLII. Maxime autem differt hæc curva a curvis algebraicis, quæ in initio A quoque habent radium osculi evanescentem. Curvarum enim algebraicarum, quæ hac indole gaudent, natura circa initium A hujusmodi formula exprimitur $x = \alpha y^m$ existente $m < 2$, attamen $m > 1$. Sit igitur $m = 2 - n$ existente n fractione unitate minore, ut sit $x = \alpha y^{2-n}$, erit $dx = \alpha(2-n) y^{1-n} dy$, ideoque $\frac{dy}{dx} = \frac{1}{\alpha(1-n)y^{1-n}} = \infty$ ob $y^{1-n} = 0$: at radius osculi, qui subnormali $\frac{ydy}{dx}$ æqualis est, erit $= \frac{y^n}{\alpha(2-n)} = 0$. Pro nostra vero curva radius osculi inventus est $= \frac{-1}{ly}$, unde radius osculi evanescens in curva algebraica quacunque erit ad radium osculi in nostræ curvæ puncto A ut $-y^n ly$ ad $\alpha(2-n)$ hoc est ut 0 ad 1; quantumvis enim exiguus sit exponens n, casu $y = 0$ semper est $y^n ly = 0$, etiamsi sit $ly = -\infty$. Quare in nostra quidem curva radius osculi in A est infinite parvus, sed tamen

men

men infinities major est, quam radius osculi evanescens in omni curva algebraica.

§. XLIII. Cognito jam initio seriei, qua valor applicatæ $PQ = q$ per abscissam $CP = p$ exprimitur, scilicet,

$$q = 1 - \tfrac{1}{2} pplp + App$$

non difficile erit hinc formam totius seriei colligere. Cum enim ex æquatione differentio-differentiali intelligatur sequentium terminorum potestates ipsius p binario crescere, valor ipsius q generatim gemina serie infinita exprimetur, eritque

$$\begin{aligned} q = 1 &+ Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c. \\ &- \alpha pplp - \beta p^4 lp - \gamma p^6 lp - \delta p^8 lp - \&c. \end{aligned}$$

in qua quidem nunc jam novimus esse $\alpha = \tfrac{1}{2}$.

§. XLIV. Cum igitur verus valor ipsius q duplici serie contineatur, ut utramque seorsim eliciamus, ponamus

$q = r - slp$ eritque differentiando

$$dq = dr - \frac{sdp}{p} - dslp$$

$$ddq = ddr - \frac{2dpds}{p} + \frac{sdp^2}{pp} - ddslp$$

Hi valores in nostra æquatione differentiali

$$p(1 - pp)\, ddq - dpdq\,(1 + pp) + pqdp^2 = 0$$

substituantur, ac termini per lp affecti seorsim nihilo æquentur, hoc modo duæ obtinebuntur æquationes:

I. q

$$\text{I. } p\,(1-pp)\,dds - (1+pp)\,dpds + psdp^2 = 0$$

$$\text{II. } p\,(1-pp)\,ddr - (1+pp)\,dpdr + prdp^2 - 2\,(1-pp)\,dpds + \frac{2sdp}{p} = 0$$

§. XLV. Ad has æquationes resolvendas ponatur

$$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$$

$$s = \alpha p^2 + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \&c.$$

eritque differentialibus sumendis

$$\frac{dr}{dp} = 2Ap + 4Bp^3 + 6Cp^5 + 8Dp^7 + \&c.$$

$$\frac{ddr}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \&c.$$

$$\frac{ds}{dp} = 2\alpha p + 4\beta p^3 + 6\gamma p^5 + 8\delta p^7 + \&c.$$

$$\frac{dds}{dp^2} = 2\alpha + 12\beta p^2 + 30\gamma p^4 + 56\delta p^6 + \&c.$$

His valoribus substitutis prima æquatio abibit in hanc:

$$\left.\begin{array}{cccccc}
2\alpha p & +12\beta p^3 & +30\gamma p^5 & +56\delta p^7 & +90\varepsilon p^9 & +\&c. \\
 & -2\alpha & -12\beta & -30\gamma & -56\delta & -\&c. \\
-2\alpha & -4\beta & -6\gamma & -8\delta & -10\varepsilon & -\&c. \\
 & -2\alpha & -4\beta & -6\gamma & -8\delta & -\&c. \\
 & +\alpha & +\beta & +\gamma & +\delta & +\&c.
\end{array}\right\} = 0$$

§. XLVI.

§. XLVI. Si jam singularum potestatum ipsius p coefficientes nihilo aequales ponantur, erit:

$$2\alpha - 2\alpha = 0; \quad \alpha \text{ manet indeterminatum}$$

$$8\beta - 3\alpha = 0; \quad \beta = \frac{1.3}{2.4}\alpha$$

$$24\gamma - 15\beta = 0; \quad \gamma = \frac{3.5}{4.6}\beta = \frac{1.3.3.5}{2.4.4.6}\alpha$$

$$48\delta - 35\gamma = 0; \quad \delta = \frac{5.7}{6.8}\gamma = \frac{1.3.3.5.5.7}{2.4.4.6.6.8}\alpha$$

$$80\varepsilon - 63\delta = 0; \quad \varepsilon = \frac{7.9}{8.10}\delta = \frac{1.3.3.5.5.7.7.9}{2.4.4.6.6.8.8.10}\alpha$$

&c. &c.

Si igitur valor coefficientis primi α constaret, quem quidem jam vidimus esse $= \frac{1}{2}$, omnes sequentes coefficientes β, γ, δ, &c. forent cogniti. Verum resolutio alterius aequationis quoque hunc nobis valorem ipsius α patefaciet.

§. XLVII. Substitutis enim seriebus ante traditis in altera aequatione proveniet;

$$\left.\begin{array}{l}
2Ap + 12Bp^3 + 30Cp^5 + 56Dp^7 + 90Ep^9 + \&c. \\
\quad - 2A - 12B - 30C - 56D - \&c. \\
-2A - 4B - 6C - 8D - 10E - \&c. \\
\quad - 2A - 4B - 6C - 8D - \&c. \\
+1 + A + B + C + D + \&c. \\
-4\alpha - 8\beta - 12\gamma - 16\delta - 20\varepsilon - \&c. \\
\quad + 4\alpha + 8\beta + 12\gamma + 16\delta + \&c. \\
+2\alpha + 2\beta + 2\gamma + 2\delta + 2\varepsilon + \&c.
\end{array}\right\} = 0$$

Unde similï modo elicitur:

$$2A - 2A + 1 - 2\alpha = 0; \text{ hinc fit } \alpha = \tfrac{1}{2}$$

$$8B - 3A - 6\beta + 4\alpha = 0; \; 2.4\, B - 1.3\, A + 2\left(2 - \frac{1.3.3}{2.4}\right)\alpha = 0$$

$$24C - 15B - 10\gamma + 8\beta = 0; \; 4.6\, C - 3.5\, B + 2\left(4 - \frac{3.5.5}{4.6}\right)\beta = 0$$

$$48D - 35C - 14\delta + 12\gamma = 0; \; 6.8\, D - 5.7\, C + 2\left(6 - \frac{5.7.7}{6.8}\right)\gamma = 0$$

$$80E - 63D - 18\varepsilon + 16\delta = 0; \; 8.10\, E - 7.9\, D + 2\left(8 - \frac{7.9.9}{8.10}\right)\delta = 0$$

§. XLVIII. Cognito igitur valore ipsius $\alpha = \frac{1}{2}$ altera series s, quæ logarithmum ipsius p involvit, tota innotescit, erit enim:

$$\alpha = \tfrac{1}{2}$$

$$\beta = \frac{1.1.3}{2.2.4}$$

$$\gamma = \frac{1.1.3.3.5}{2.2.4.4.6}$$

$$\delta = \frac{1.1.3.3.5.5.7}{2.2.4.4.6.6.8}$$

$$\varepsilon = \frac{1.1.3.3.5.5.7.7.9}{2.2.4.4.6.6.8.8.10}$$

&c.

fietque hinc $s = \alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \&c.$

§. XLIX. Quod autem ad alteram seriem attinet

$$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + \&c.$$

primus

primus coefficiens A hinc manet indeterminatus, cujus rei ratio eft, quod has feries ex æquatione differentiali fecundi gradus elicuimus, quæ duplici determinatione indiget, ut ad noftrum cafum accommodetur. Quare valorem hujus coefficientis A ex ipfa curvæ natura definiri oportet, eo autem invento, reliqui innotefcent ex his formulis, ad quas fuperiores redeunt.

$$B = \frac{1\cdot 3}{2\cdot 4} A - \tfrac{1}{2} \alpha \left(\frac{3}{2\cdot 2} + \frac{1}{1\cdot 1}\right)$$

$$C = \frac{3\cdot 5}{2\cdot 4} B - \tfrac{1}{4} \beta \left(\frac{3}{3\cdot 3} + \frac{1}{2\cdot 2}\right)$$

$$D = \frac{5\cdot 7}{6\cdot 8} C - \tfrac{1}{6} \gamma \left(\frac{3}{4\cdot 4} + \frac{1}{4\cdot 4}\right)$$

$$E = \frac{7\cdot 9}{8\cdot 10} D - \tfrac{1}{8} \delta \left(\frac{3}{5\cdot 5} + \frac{1}{4\cdot 4}\right)$$

§. L. His autem omnibus coefficientibus inventis ad datam quamvis abfciffam $CP = p$, valor refpondentis applicatæ PQ $= q$ ita definitur, ut fit

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$$
$$- \alpha pp\, lp - \beta p^4\, lp - \gamma p^6\, lp - \delta p^8\, lp - \&c.$$

quæ feries fi abfciffa p fuerit unitate multo minor, fatis prompte convergit, ut inde valor ipfius q cognofci queat. Hinc vero etiam applicatæ, quæ abfciffis multo majoribus unitate refpondent, definiri poterunt, quia abfciffæ $\frac{1}{p}$ refpondet applicata $\frac{q}{p}$. Quare fi abfciffa unitate multo major ponatur $= P$ eique refpondens applicata $= Q$ ob $p = \frac{1}{P}$ & $q = pQ = \frac{Q}{P}$ erit

 Q=

$$Q = P + AP^{-1} + BP^{-3} + CP^{-5} + DP^{-7} + \&c.$$
$$+ \alpha P^{-1}\, lP + \beta P^{-3}\, lP + \gamma P^{-5}\, lP + \delta P^{-7}\, lP + \&c.$$

Hinc si abscissa P fiat infinita erit

$$Q = P + \frac{\alpha lP}{P} \text{ seu } Q - P = \frac{\alpha lP}{P}$$

unde natura rami Dq in infinitum extensi & ad asymtotam CV appropinquantis colligitur.

§. LI. Quia porro novimus, si $p = 1$ fore $q = \frac{\pi}{2}$ pro hoc casu aequatio inventa hanc formam ob $l1 =$ induet

$$\frac{\pi}{2} = 1 + A + B + C + D + E + \&c.$$

Cum igitur valor A nondum sit definitus, reliqui vero B, C, D &c. ab eo pendeant, haec aequatio conditionem continet, qua valor ipsius A determinatur. Ita scilicet valorem ipsius A comparatum esse oportet, ut summa seriei infinitae $1 + A + B + C + \&c.$ fiat $\frac{\pi}{2}$. Verum si valores reliquarum litterarum B, C, D &c. qui ab A pendent, evolvantur, tam complicatae resultant expressiones, ut hinc valor ipsius A neutiquam erui possit.

§. LII. Ad hanc constantem A determinandam alia patet via, si datae cujuspiam ellipsis perimeter ex altera formula in numeris fuerit inventa. Quae methodus cum requirat, ut omnes coefficientes in fractionibus decimalibus evolvantur, computo peracto reprerietur:

$\alpha =$

$\alpha = 0,5000000000$ A quaeritur

$\beta = 0,1875000000;\ B = 0,3750000000\,A - 0,1093750000$

$\gamma = 0,1171875000;\ C = 0,2343750000\,A - 0,0820312500$

$\delta = 0,0854492188;\ D = 0,1708984375\,A - 0,0641886393$

$\varepsilon = 0,0672912598;\ E = 0,1345825195\,A - 0,0524978638$

$\zeta = 0,0555152893;\ F = 0,1110305786\,A - 0,0443481445$

$\eta = 0,0472540855;\ G = 0,0945081711\,A - 0,0383663416$

$\theta = 0,0411363691;\ H = 0,0822727381\,A - 0,0337966962$

$\iota = 0,0364228268;\ I = 0,0728456536\,A - 0,0301949487$

$\varkappa = 0,0326793696;\ K = 0,0653587392\,A - 0,0272843726$

&c.

Hisque valoribus inventis, si abscissa sit $CP = p$, valor applicatae q ita definietur ut sit:

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12} + Gp^{14} + Hp^{16} + Ip^{18} + Kp^{20} + \&c.$$
$$- pp\,lp.(\alpha + \beta p^2 + \gamma p^4 + \delta p^6 + \varepsilon p^8 + \zeta p^{10} + \eta p^{12} + \theta p^{14} + \iota p^{16} + \varkappa p^{18} + \&c.)$$

§. LIII. Deinde vero supra ejusdem applicatae q valorem ita invenimus expressum ut sit:

$$q = \frac{\pi}{2}\left(1 - \frac{1.1}{2.2}(1-pp) - \frac{1.1.1.3}{2.2.4.4}(1-pp)^2 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}(1-pp)^3 - \&c.\right)$$

Nunc igitur ex utraque formula pro eodem quopiam valore re ipsius p eruamus valorem ipsius q, ut deinceps ex aequalitate horum duorum elicere queamus valorem coefficientis A. Pro p vero non nimis exiguam fractionem substitui conveniet, ne expressio posterior nimis lente convergat, tam parvum tamen assumamus ut coefficientes pro superiore forma computati valori q ad 10 figuras inveniendo sufficiant.

§. LIV. Ponamus ergo ad commodum calculi $p = \frac{1}{5}$, erit in logarithmis hyperbolicis:

$$-lp = 1,60943791243$$

Jam vero est

$$
\begin{array}{lcll}
\alpha pp & = & 0,02000000000 & \\
\beta p^4 & = & 0,00030000000 & \\
\gamma p^6 & = & 0,00000750000 & \\
\delta p^8 & = & 0,00000021875 & \\
\varepsilon p^{10} & = & 0,00000000689 & \\
\zeta p^{12} & = & 0,00000000023 & \\
\eta p^{14} & = & 0,00000000001 & \\
\hline
 & & 0,02030772588 & \text{coefficiens ipsius } -lp \\
 & & 1,60943791243 & \\
\hline
 & & 0,03268402394 & \text{productum.}
\end{array}
$$

Deinde est

Ap

$$
\begin{array}{lcll}
Ap^{2} & = & 0{,}04000000000\,A & \\
Bp^{4} & = & 0{,}00060000000\,A & -\ 0{,}00017500000 \\
Cp^{6} & = & 0{,}00001500000\,A & -\ 0{,}00000525000 \\
Dp^{8} & = & 0{,}00000043750\,A & -\ 0{,}00000016432 \\
Ep^{10} & = & 0{,}00000001378\,A & -\ 0{,}00000000538 \\
Fp^{12} & = & 0{,}00000000045\,A & -\ 0{,}00000000016 \\
Gp^{14} & = & 0{,}00000000002\,A & -\ 0{,}00000000001 \\
\hline
 & & 0{,}04061545175\,A & -\ 0{,}00018041987
\end{array}
$$

Ex his conficitur

$$q = 0{,}04061545175\,A + 1{,}03250325036040 7$$

§. LV. Nunc eundem valorem ipsius q ex altera aequatione quaeramus, & cum sit $p = \frac{1}{5}$, erit $1 - pp = \frac{24}{25}$ sit $nn = \frac{24}{25}$ erit

$$q = \frac{\pi}{2}\left(1 - \frac{1.1}{2.2}nn - \frac{1.1.1.3}{2.2.4.4}n^{4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}n^{6} - \&c.\right)$$

ponatur ad abbreviandum

$$q = \frac{\pi}{2} - \mathfrak{A}n^{2} - \mathfrak{B}n^{4} - \mathfrak{C}n^{6} - \mathfrak{D}n^{8} - \mathfrak{E}n^{10} - \&c.$$

Verum hoc casu ob $nn = \frac{24}{25}$ series ista nimis lente convergit, quam ut hinc valor ipsius q satis exacte elici queat, quare ut utrinque parem convergentiam obtineamus ponamus $p = \frac{1}{\sqrt{2}}$, ut sit tam $pp = \frac{1}{2}$ quam $nn = \frac{1}{2}$; calculum vero tantum ad 6 figuras expediamus: eritque

$App = 0,500000\,A$

$Bp^{4} = 0,093750\,A - 0,027344$

$Cp^{6} = 0,029297\,A - 0,010254$

$Dp^{8} = 0,010681\,A - 0,004012$

$Ep^{10} = 0,004206\,A - 0,001640$

$Fp^{12} = 0,001735\,A - 0,000693$

$Gp^{14} = 0,000738\,A - 0,000300$

$Hp^{16} = 0,000321\,A - 0,000132$

$Ip^{18} = 0,000142\,A - 0,000059$

$Kp^{20} = 0,000064\,A - 0,000026$

Summa reliq: $60\,A - 24$

Summ. $0,640994\,A - 0,044484 + 0,320497.\ l\dfrac{1}{p} + 1$

ergo $q = 1,066592 + 0,640994\,A$

at altera expressio dat $q = 1,350647$, unde fit

$$A = \frac{284055}{640994} = 0,443147$$

§. LVI. Quanquam hic valor non ultra 6 figuras extenditur, tamen casui non tribuendum videtur, quod iste numerus inventus $0,443147$ a logarithmo binarii $0,69314718$ unitatis quadrante $0,25$ praecise deficiat. Quae conjectura si veritati esset consentanea, valorem litterae A ad plurimas figuras exhibere liceret, cum enim sit

$l2 = 0,693147180559945309417232l$

foret $A = l2 - \frac{1}{4}$ ideoque

$A = 0,443147180559945309417232l.$

Quod

Quod autem valor coefficientis hujus A ſit revera $= la - \frac{1}{2}$, ſequenti modo demonſtro, hincque conjecturam confirmo.

§. LVII. Comparo ſcilicet arcum ellipticum AYP, cujus ſemiaxes $AC = 1$, $CP = p$ cum arcu parabolico AZS ſuper eodem axe AC deſcripto, qui in A cum ellipſi communem habeat curvaturam. Sumta abſciſſa communi $AX = x$, ſit applicata ellipſis $XY = y$ & parabolæ $XZ = z$, erit $y = p\sqrt{(2x - xx)}$ & $z = p\sqrt{2x}$, ideoque $dy = \frac{pdx(1-x)}{\sqrt{(2x-xx)}}$ & $dz = \frac{pdx}{\sqrt{2x}}$: unde ſit

arcus ellipticus $AY = \int dx\sqrt{\left(1 + \frac{pp(1-x)^2}{2x - xx}\right)}$

arcus parabolicus $AZ = \int dx\sqrt{\left(1 + \frac{pp}{2x}\right)}$. Conſtat autem

eſſe $AZ = x\sqrt{\left(1 + \frac{pp}{2x}\right)} + \frac{1}{4}pp\, l\, \frac{\sqrt{\left(1+\frac{pp}{2x}\right)} + 1}{\sqrt{\left(1+\frac{pp}{2x}\right)} - 1}$: Hinc ſi ponatur

$x = 1$, erit arcus parabolicus $AZS = \sqrt{(1 + \frac{1}{2}pp)} + \frac{1}{4}pp\, l\, \frac{\sqrt{(1+\frac{1}{2}pp)} + 1}{\sqrt{(1+\frac{1}{2}pp)} - 1}$

At in formulis integralibus erit

$$\sqrt{\left(1 + \frac{pp(1-x)^2}{2x - xx}\right)} = \sqrt{\left(1 + \frac{pp}{2x} - \frac{pp(3-2x)}{4-2x}\right)}$$

Quia autem comparationem non ad altiores ipſius p poteſtates extendere opus eſt quam ad ſecundam: coefficientes enim altiorum ipſius p poteſtatum ex minoribus jam definivimus, rejectis terminis, qui continent p^4 & altiores poteſtates, erit

$\sqrt{(1}$

$\sqrt{(1+\frac{pp(1-x)^2}{2x-xx})} = \sqrt{(1+\frac{pp}{2x}-\frac{pp(3-2x)}{2(2-x)})}$ ideoque

$AY = \int dx \sqrt{(1+\frac{pp}{2x})} - \frac{1}{2} pp \int \frac{dx(3-2x)}{2-x}$, Integralibusque actu sumtis

$$AY = x\sqrt{(1+\frac{pp}{2x})} + \frac{1}{2} ppl \frac{\sqrt{(1+\frac{pp}{2x})}+1}{\sqrt{(1+\frac{pp}{2x})}-1} - \frac{1}{2} ppx - \frac{1}{2} ppl \frac{2-x}{2}$$

Poniatur jam $x = 1$, ut prodeat arcus $AYP = q$, erit
$q = \sqrt{(1+\frac{1}{2} pp)} + \frac{1}{4} ppl(\sqrt{(1+\frac{1}{2} pp)}+1) - \frac{1}{4} ppl(\sqrt{(1+\frac{1}{2} pp)} - 1) - \frac{1}{2} pp + \frac{1}{4} ppl2.$

§. LVIII. Jam quoniam ad altiores ipsius p potestates non respicimus, erit $\sqrt{(1+\frac{1}{2} pp)} = 1 + \frac{1}{4} pp$, unde fiet

$$q = 1 + \frac{1}{4} pp + \frac{1}{4} pp\, l(2+\frac{1}{4} pp) - \frac{1}{4} pp\, l\frac{1}{4} pp - \frac{1}{2} pp + \frac{1}{4} ppl2$$

ubi pro $l(2+\frac{1}{4} pp) = l2 + \frac{1}{8} pp$ scribere licet $l2$, ita ut sit $q = 1 - \frac{1}{4} pp + \frac{1}{2} ppl2 - \frac{1}{2} pplp + \frac{1}{2} ppl2$

seu $q = 1 - \frac{1}{2} pplp + pp(l2 - \frac{1}{4})$

unde perspicitur coefficientem ipsius pp, quem ante litera A indicavimus

vimus esse $= l2 - \frac{1}{4}$, omnino uti ex casu ante commemorato conjectura sumus consecuti.

§. LIX. Pro curva igitur initio proposita $AQDq$, si fuerit abscissa $CP = p$ & applicata $PQ = q$, erit Fig. 1.

$$q = 1 + App + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + \&c.$$
$$-(\alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \&c.)\, lp$$

ubi coefficientes ita determinantur:

$$\begin{array}{l|l}
A = l2 - \frac{1}{4} & \alpha = \frac{1}{2} \\
B = \frac{1.3}{2.4} A - \frac{1}{2}(\alpha - \beta) + \frac{1}{2}\cdot\frac{\beta}{2} & \beta = \frac{1.3}{2.4}\,\alpha \\
C = \frac{3.5}{4.6} B - \frac{1}{3}(\beta - \gamma) + \frac{1}{4}\cdot\frac{\gamma}{3} & \gamma = \frac{3.5}{4.6}\,\beta \\
D = \frac{5.7}{6.8} C - \frac{1}{4}(\gamma - \delta) + \frac{1}{6}\cdot\frac{\delta}{4} & \delta = \frac{5.7}{6.8}\,\gamma \\
E = \frac{7.9}{8.10} D - \frac{1}{5}(\delta - \varepsilon) + \frac{1}{8}\cdot\frac{\varepsilon}{5} & \varepsilon = \frac{7.9}{8.10}\,\delta \\
F = \frac{9.11}{10.12} E - \frac{1}{6}(\varepsilon - \zeta) + \frac{1}{10}\cdot\frac{\zeta}{6} & \zeta = \frac{9.11}{10.12}\,\varepsilon
\end{array}$$

&c.

series haec valde convergit, si abscissa p fuerit fractio valde parva, sin autem sit unitate multo major, iisdem manentibus coefficientibus erit

$$q = p + \frac{A}{p} + \frac{B}{p^3} + \frac{C}{p^5} + \frac{D}{p^7} + \frac{E}{p^9} + \&c.$$
$$+ \left(\frac{\alpha}{p} + \frac{\beta}{p^3} + \frac{\gamma}{p^5} + \frac{\delta}{p^7} + \frac{\varepsilon}{p^9} + \&c. \right) lp$$

§. LX. Verum si abscissa p non multum ab unitate discrepet, uti conveniet hac serie supra §. XXVI. inventa

$$q = 1 + pp \left\{ \begin{array}{l} \left(\frac{\pi}{2} - 1\right) + \left(\frac{1.3}{2.2}\cdot\frac{\pi}{2} - 1\right)(1-pp) + \\ \left(\frac{1.3.3.5}{2.2.4.4}\cdot\frac{\pi}{2} - 1\right)(1-pp)^2 + \left(\frac{1.3.3.5.5.7}{2.2.4.4.6.6}\cdot \right. \\ \left. \frac{\pi}{2} - 1\right)(1-pp)^3 + \&c. \end{array} \right.$$

quae etiam ex natura ellipsis in hanc convertitur

$$q = p + \frac{1}{p} \left\{ \begin{array}{l} \left(\frac{\pi}{2} - 1\right) - \left(\frac{1.3}{2.2}\cdot\frac{\pi}{2} - 1\right)\frac{(1-pp)}{pp} + \\ \left(\frac{1.3.3.5}{2.2.4.4}\frac{\pi}{2} - 1\right)\frac{(1-pp)^2}{p^4} - \left(\frac{1.3.3.5.5.7}{2.2.4.4.6.6} \right. \\ \left. \frac{\pi}{2} - 1\right)\frac{(1-pp)^3}{p^6} + \&c. \end{array} \right)$$

unde prout fuerit vel $p > 1$ vel $p < 1$ eam eligere licet, cujus termini vel iisdem signis procedant, vel alternantibus. Plerumque autem praestat ad summam proxime definiendam signa eligere, alternantia.

Pro-

Problema.

§. LXI. Datis axibus conjugatis ellipsis, in numeris proxime exhibere ejus perimetrum.

Solutio.

Sint semiaxes ellipsis 1 & p, & quadrans perimetri $= q$, atque per formulas inventas valor ipsius q in numeris definiri poterit, dummodo ea eligatur, cujus termini maxime convergant. Quatuor autem adepti sumus formulas quae sunt:

$$\text{I. } q = 1 + App + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12} \text{ \&c.}$$
$$- (\alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \epsilon p^{10} + \zeta p^{12} \text{ \&c.})\, lp$$

$$\text{II. } q = p + A\frac{1}{p} + B\frac{1}{p^3} + C\frac{1}{p^5} + D\frac{1}{p^7} + E\frac{1}{p^9} + F\frac{1}{p^{11}} + \text{\&c.}$$
$$+ \left(\frac{\alpha}{p} + \frac{\beta}{p^3} + \frac{\gamma}{p^5} + \frac{\delta}{p^7} + \frac{\epsilon}{p^9} + \frac{\zeta}{p^{11}} + \text{\&c.}\right) lp$$

$$\text{III. } q = 1 + pp\left(\mathfrak{A} + \mathfrak{B}(1-pp) + \mathfrak{C}(1-pp)^2 + \mathfrak{D}(1-pp)^3 + \mathfrak{E}(1-pp)^4 + \text{\&c.}\right)$$

$$\text{IV. } q = p + \frac{1}{p}\left(\mathfrak{A} - \mathfrak{B}\frac{(1-pp)}{pp} + \mathfrak{C}\frac{(1-pp)^2}{p^4} - \mathfrak{D}\frac{(1-pp)^3}{p^6} + \mathfrak{E}\frac{(1-pp)^4}{p^8} - \text{\&c.}\right)$$

Horum

Horum autem tergeminorum coefficientium valores sunt in numeris:

$A = 0,44314718056$	$\alpha = 0,50000000000$	$\mathfrak{A} = 0,57079632679$
$B = 0,05680519271$	$\beta = 0,18750000000$	$\mathfrak{B} = 0,17809724510$
$C = 0,02183137044$	$\gamma = 0,11718750000$	$\mathfrak{C} = 0,10446616728$
$D = 0,01154452143$	$\delta = 0,08544921875$	$\mathfrak{D} = 0,07378655152$
$E = 0,00714200029$	$\varepsilon = 0,06729125977$	$\mathfrak{E} = 0,05790863665$
$F = 0,00485474337$	$\zeta = 0,05551527931$	$\mathfrak{F} = 0,04643855029$
$G = 0,00351468795$	$\eta = 0,04725408554$	$\mathfrak{G} = 0,03917161592$
$H = 0,00266223578$	$\theta = 0,04113636911$	$\mathfrak{H} = 0,03386971991$
$I = 0,00208639732$	$\iota = 0,03642282682$	$\mathfrak{I} = 0,02983116632$
$K = 0,00167916842$	$\kappa = 0,03267936962$	$\mathfrak{K} = 0,02665267507$
		$\mathfrak{L} = 0,02408604339$

Hinc pro quavis ellipsis specie habebitur series convergens, unde ejus perimeter definiri poterit, veluti

si ponatur $p = \frac{1}{10}$ erit $q = 1,01599354021$

si sit $p = \frac{1}{5}$ erit $q = 1,05050222700$

si sit $p = \frac{1}{\sqrt{2}}$ erit $q = 1,3506$[illegible]